

Numerical Modeling of Laterally Loaded Piles in Sandy Soils

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Abstract. Pile foundations are more common than any other type of deep foundations. With the rapid growth of metropolitan areas and fast industrialization resulting from the fast-paced economic globalization, requirement of construction of heavier and taller structures on marginal site has become inevitable. This paper presents an experimental investigation on the lateral load carrying capacity of model bamboo pipe piles of different slenderness (L/d 25, 30 and 38) ratios embedded in loose layer in-between dense layers soil strata to determine ultimate lateral displacement by experimentally. Also, two-dimensional finite element parametric analyses with the help of SoilWorks 2D software have been carried out to investigate the influence of lateral loads on the lateral performance of pile groups in sandy soils. Different pile groups and soil densities were considered to examine the salient features of this complex soil–structure interaction problem. Numerical results show that the lateral response of pile groups embedded in sands is influenced by the soil layer, the slenderness ratios (L/d), the lateral loads as well as the soil density. When the lateral load acts in the different soil layers, the lateral response of the piles increases with the pile groups as well as the sand density. On the other hand, the lateral response of piles embedded in a loose layer in-between dense sand layer the lateral load increases greatly with the slenderness ratio and does not vary with its pile stiffness. The variation for values of lateral displacement, bending moment with depth of pile is observed to be non-linear in nature. Data were taken during the lateral loading of three 450, 500 and 550mm diameter test piles installed at a site where the soils consisted of clean fine sand to silty fine sand. With theoretical studies as a basis, a method was devised for predicting the family of p-y curves based on the properties of sand and pile deformations. The procedure was employed for predicting p-y curves at the experimental site and computed results are compared with experimental results. The agreement is good.

Keywords: Pile groups, Lateral response, Lateral loads, Finite element analysis modeling, SoilWorks 2D software.

1 Introduction

Pile foundations are more common than any other type of deep foundations. With the rapid growth of metropolitan areas and fast industrialization resulting from the fast-paced economic globalization, requirement of construction of heavier and taller structures on marginal site has become inevitable. Loads due to ship thrust on offshore structures, lateral loads due to earthquakes, bomb blasts are examples of dynamic loadings. To understand the deformation behaviour of each of the pile in a pile group, subjected to lateral loads or a combination of vertical and lateral loads, it is first necessary to have a clear idea of the deformation behaviour of single pile of similar batter under lateral loads. In the above examples, there are some cases in which the external horizontal loads act at the pile head. Such loading is called active loading [10]. Consequently, the response of a single pile differs from that of a pile placed within a pile group: Ilyas et al. and McVay et al. [6,7]. Each pile in a group, whether loaded axially or laterally, generates a displacement field of its own around itself. The displacement field of each pile interferes and overlaps with those of the adjacent piles; this results in the interaction between piles. In a laterally loaded pile group, each pile pushes the soil in front of it. Movement of the piles placed in the first (leading) row in the direction of the applied force is resisted by the soil in front of it. In contrast, the piles in the rows behind the first row (i.e., the piles in the trailing rows) push on the soil which in turn pushed on the piles in the rows in front of them. The resistive forces acting on the trailing-row piles are in general less than the resistive forces acting on the leading row: Ilyas et al. and Salgado [6,17].

2 Literature Review

During the last few decades, many researchers have studied the behaviour of laterally loaded piles using both laboratory tests and theoretical studies. Biswas et al. [1] carried out experimental investigation of free-head model piles under lateral load in homogenous and layered sand. The experimental study was supplemented by numerical study to determine co-efficient of horizontal modulus of sub-grade reaction (C_h). Byung, Nak-Kyung et al.[2] have observed that, testing of the pile embedded in Nak-Dong river sand, located in south Korea, under monotonic lateral loadings. The lateral loads and the soil-pile reactions of driven piles increase as the driving energies increase. Salini and Girish [16] concluded that, the lateral-load capacity of pile group increases as the density of sand increases for the same slenderness ratio. The lateral-load capacity of pile increases with increase in length for same diameter hence passive resistance was mobilized to increase the embedment length of a pile. Georgiadis [5] carried out three-dimensional finite element analyses using PLAXIS 3D and presented the response of laterally loaded piles of different geometries, installed at several distances from slopes of various

inclinations. From analysis, p - y curves are developed for the case of undrained lateral loading of piles near the crest of clay slopes and implemented the p - y curve into a commercial sub-grade reaction computer code to perform a series of parametric numerical analyses. Muthukkumaran [8] investigated that, the lateral load pile capacity of the pile, lateral load-lateral displacement response of the pile at pile head, effect of slopes and embedment length on pile capacity and bending-moment (BM) profile along the pile shaft were studied. William Higgins et al. [18] analysed laterally loaded piles embedded in single-layer elastic soil with constant and linearly varying modulus and in two-layer elastic soil with constant modulus within each layer using the Fourier Finite Element Method. Based on the analysis, it is observed that pile response is a function of the relative stiffness of pile and soil, and of the pile slenderness ratio. Soil layering is an important factor that affects laterally loaded pile response by Basu and Salgado [3]. Layering has been taken into account approximately in some pile analyses by either assuming a linear variation of k with depth or by proposing different p - y curves for different soil depths. Rathod et al. [11] have investigated that, the effect of slope on soil reaction-lateral displacement (p - y) curves for laterally loaded piles in soft clay. The results show that the pile top displacement and the bending moment (BM) in the pile decrease with an increase in the slope. Sawant and Shukla [15] carried out a three-dimensional finite element analysis to investigate the effect of edge distance from the slope crest of a laterally loaded pile embedded in clayey soil of soft to medium consistency in the sloping ground for different slope angles and pile lengths. The results indicated that the pile top displacement and the bending moment in the pile decrease with an increase in the edge distance from the slope crest. From the literature, it is clear that only a few limited research works have been carried out on piles subjected to lateral load in layered cohesionless soil, and the behaviour of pile embedded in layered cohesionless soils requires further study. However very little work has been reported on the numerical studies of laterally loaded piles, barring a few theoretical and laboratory test results. In this paper a nonlinear finite element model has been developed to study the behaviour of bamboo pipe piles under static lateral loads, using the developed numerical model.

3 Experimental Investigation

The main objective of the experimental research program presented in this section is threefold. The first objective is to develop a model of pile and foundation soil in the laboratory. The second objective is to investigate the ultimate lateral capacity of a single bamboo pipe pile at ground surface of a free head in cohesionless soil. The third objective is to study the behaviours of different pile group with different slenderness (L/d) ratios 25, 30 and 38 with 3D pile spacing in loose layer in-between dense layers soil medium to withstand lateral displacements induced by experimentally. The detail experimental investigation is carried out below;

3.1 Experimental Set-up

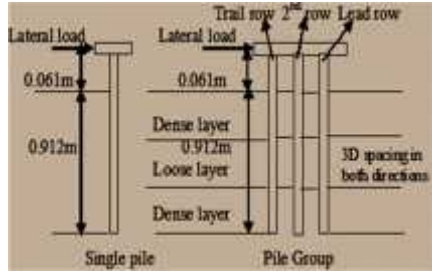


Fig.1. Layouts of single and bamboo pile group. **Fig.2** Different slenderness ratios pile groups of bamboo pipe materials model

The purpose of experimental investigation is prototype reduced to a model scale. For example, a 1/15 (1/N) scaled model would require that a prototype pipe pile of 14.60m long by 0.36m circular diameter modelled by bamboo pipe pile of 0.973m long (overall length) and 24mm external diameter with 2mm wall thickness was used as a model pile (prototype dimension/N). Figure 1 is the layouts of single and pile group of the model, which was modelled in the experimental investigation at 1/15 scale.

The Young's modulus ($E_m=1.617 \times 10^8 \text{ kN/m}^2$) and the moment of inertia of the model pile (I_m) determined as $4.787 \times 10^{-9} \text{ m}^4$ and Poisson's ratio (μ) as 0.30. The bending stiffness, $E_m I_m$, of 0.77405 kN-m^2 . The dimension of test tank is decided based on the

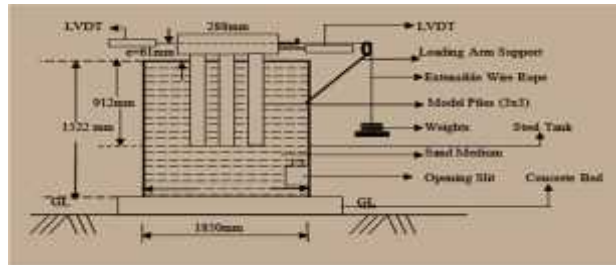


Fig.3. Experimental set-up for lateral load tests.

influence zone of soil mass from pile. It is 10 times the pile diameter in the direction of loading for piles under static lateral load according to Narasimha Rao et al. and Poulos [9,10]. Hence, the static lateral load tests were conducted in a test tank with a dimension of 1850mm x 1850mm x 1522mm placed on a loading platform. The static lateral load is applied by means of dead weights (slotted type) placed on a hanger connected to a flexible steel wire, strung over a frictionless pulley supported by a loading platform as shown in figure 3.

Soil used in the Experimental Studies

A clean, dry sand (Indian standard sieve through 1.18mm passing and 75 μ retained) was used as the foundation soil in this study. The specific gravity of sand was found to be 2.67. The minimum and maximum dry unit weights of sand were found to be 16.00 and 19.90kN/m³, respectively. The particle size distribution was determined using the dry sieving method, and the uniformity coefficient (c_u) and coefficient of curvature (c_c) for the sand were 2.41 and 1.20, respectively. The laboratory model tests were conducted on sand with maximum and minimum void ratios 0.637 and 0.316, for loose sand and dense sand respectively. The relative densities of the sand were 30% and 90%, respectively, and the angles of internal friction were 31⁰ and 36⁰, respectively.

Experimental Procedure

Here soil medium of loose layer in-between dense layers have been used to carry out the experiment. Bamboo pipe piles were used as the model pile in the experimental set up. In soil layer external lateral load is applied on the model pile embedded in the cohesionless soil with a depth of 0.456m. The depth of soil was calculated using H/D ratio of 0.50. i.e., $H=D \times 0.50=0.912 \times 0.50=0.456$ m. The top and bottom sand layers depth were calculated to be 0.228m each. Using sand raining technique from the height of 600mm from bottom of tank the sand is filled into the tank to get dense state. The model piles were placed in their positions at the top of the bearing stratum (dense sand layer). The middle layer is filled with the sand from a height of fall 10mm to get loose state; remaining top layer is filled by sand raining technique from a height of 600mm to get dense state. For slenderness (L/d) ratio 25, 30 and 38, the embedment length would be 600, 720 and 912mm respectively, from the pile toe. The lateral load is applied at pile head (61mm above the ground surface). For each increment of lateral load, the lateral displacement of the pile was measured at pile head using LVDT instrument with display unit. When the lateral displacement of the pile ceases, the next lateral load increment was applied. The lateral load was applied till the lateral displacement reaches 10.50% of pile diameter (0.105d) and the corresponding load was taken as allowable lateral load capacity of the pile according to Narasimha Rao et al. (1998) and Chandrasekaran et al. [4,9].

4 Finite Element Model

An important objective of this research is to determine bending moment, shear force, soil reaction and lateral load-lateral displacement behaviour of bamboo pipe piles can be performed using the numerically.

4.1 Pile-Soil Models and Parameters

The interactions between the foundation soil and the piles would be the best modeled by a finite element program capable of solving two-dimensional problems. However, 2D finite element analysis would require more time and effort. To give some understanding of the complex interactions between foundation soil and piles it was decided to use the computer program SoilWorks 2D for numerical investigation. The interactions between the foundation soil and the piles can be best obtained by using 2-dimensional finite element model software. Description of the capabilities of SoilWorks 2D are presented below. SoilWorks 2013(v2.1) is all-in-one 2D Finite element analysis and analytical software for structural and geo-technical engineers. SoilWorks is fully integrated pre/post and solve, complete FEM Software package, CAD based environment, intuitive, automation and robust.

The input parameters used in this analysis are presented in Table 1.

Table 1. Pile properties

Parameters	Notation	Bamboo pipe pile
Material model	----	Linear elastic
Element type	----	2D 3node triangular element (beam element).
Diameter (m)	d_o	0.024
	d_i	0.020
Shape	----	Pipe
Material type	----	bamboo
Modulus of elasticity (kN/m ²)	E	1.617X10 ⁸
Poisson's Ratio	μ	0.30
Unit weight (kN/m ³)		04.00
	L/d=25	0.600
Pile length (m)	L/d=30	0.720
	L/d=38	0.912

An embedded pile consists of beam elements with special interface elements providing the interaction between the beam and the surrounding soil. The beam elements are considered as linear elastic and its behaviour is defined using elastic stiffness properties. The behaviour of interfaces for the modeling of soil-pile interaction is treated with elastic-plastic model. The beam element is three-node line elements with six degrees of freedom per node, three translational degrees of freedom (u_x , u_y , and u_z) and three rotational degrees of freedom (θ_x , θ_y , and θ_z). In the present study, the pile is modeled as embedded pile having free connection at it's top. The material parameters of the embedded pile distinguish between the parameters of beam and parameters of skin resistance and foot resistance. The properties of material used in analysis are presented in Table 2.

Table 2. Material properties used in the analyses

Parameters	Name	Dummy soil	Dense sand	Loose sand
Material model	Model	Mohr-coulomb		
Material behaviour	Type	Drained		
Unsaturated unit weight (kN/m ³)	unsat	0.001	19.90	16.00
Saturated unit weight (kN/m ³)	sat	0.001	21.00	18.00
Young's modulus (kN/m ²)	E	0.010	21000	15000
Poisson's Ratio	μ	0.005	0.30	0.40
Cohesion (kN/m ²)	C	0.10	1	1
Friction angle ($^{\circ}$)		1	36	31
Material type			Sandy soil (Rees et al.)	
Horizontal reaction (kN/m ³)	K _h	0.271	16300	7872
Strain at 50% stress	---	---	----	----
Unit ultimate skin friction (kN/m ²)	----	0.0069	40	21
Unit ultimate bearing capacity (kN/m ²)	q _u	0.0069	4000	600

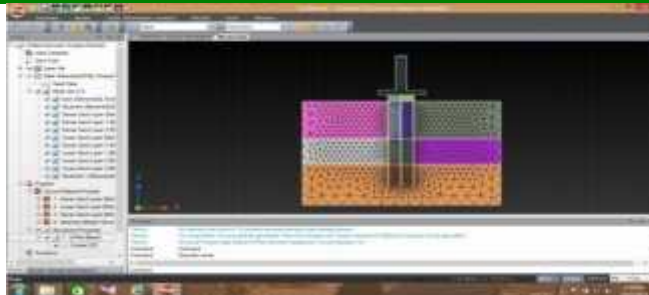


Fig.3. 2D Finite element model of soil-pile-with pile raft structure for nine bamboo pipe pile group

Using the surfaces assigned with material properties, mesh is generated in SoilWorks 2D software. Figure 3 shows the typical discretization of 2D finite element model of soil-pile-with pile raft structure for nine bamboo pipe pile group in loose layer in-between dense layers at an eccentricity of 61mm above the ground level for soil model of slenderness ratio (L/d) 38. The program contains default p-y curves that can be used for different types of soils. As an alternative, the program also allows the user to input p-y curves developed using other formulations. For the analyses carried out in this research the piles were discretized into 100 elements in the SoilWorks 2D program.

4.2 Validation of the Proposed Numerical Model

The computer code developed for nonlinear static analysis was validated for model piles: Ranjan et al. [12].

Model Pile : Ranjan et al. [12]

A model aluminum pile embedded in soft clay was tested : Ranjan et al. [12] at a horizontal load of 24.50N at the pile head. A nonlinear static analysis was performed on the pile. The pile and soil parameters used in the analysis are given in Table 3.

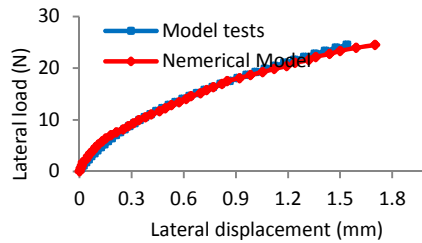


Fig.4. Comparison of lateral load-lateral displacement for vertical pile with experimental results of Ranjan et al. [12].

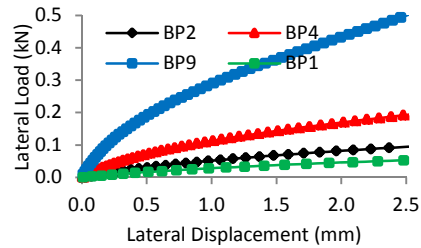


Fig.5. Comparison between single pile and pile group lateral load results of bamboo pipe piles as slenderness ratio as 25.

Table 3. Pile and soil parameters: Ranjan et al.[12]

Parameters	Details
Soil	Liquid limit=54%, plastic limit=25%, consistency index=0.48, undrained shear strength $C_u=15.2\text{kPa}$, soil modulus $E=600\text{kPa}$, unit weight of soil $=18\text{kN/m}^3$, water content=40%.
Pile	Diameter $D=9.5\text{mm}$, wall thickness=1mm, embedded length of pile=360mm.

The comparison of results for the vertical piles are shown in Figure 4. The numerical results show a fairly good comparison with the experimental results. Hence, it could be concluded that the numerical scheme adopted in the present investigation is capable of modeling the behaviour of vertical piles under lateral loads.

5 Experimental and Numerical Results and Discussions

5.1 Lateral Load-Displacement Behaviour of Piles

Figure 5 shows a typical experimental lateral load lateral displacement curves for $L/d=25$ for different bamboo pipe pile configuration (single pile, two piles, four piles and nine piles). It is observed that when number of piles increases from single pile to nine piles, the behaviour of pile is almost nonlinear. It shows very clearly that at 2.5mm lateral displacement, the ultimate lateral load capacity increases from 0.049kN, 0.091kN, 0.193kN and 0.494kN by single pile, two piles, four piles and nine piles respectively at 3D pile spacing.

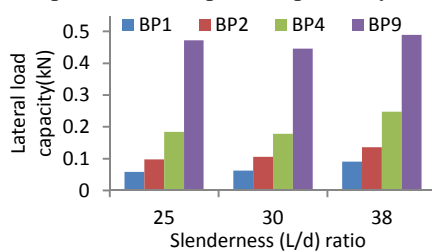


Fig.6. Shows a variation of the lateral load capacity of pile with different slenderness ratios in loose layer in-between dense layers.

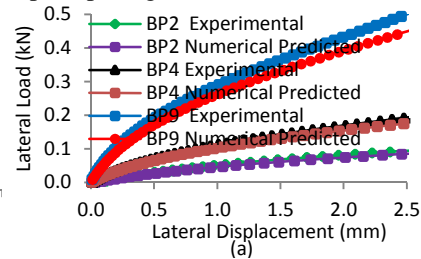


Fig.7. Comparison between the experimental and predicted lateral load results at 3D pile spacing in loose layer in between dense layers as $SL=25$.

From experimental work figure 6, it is concluded that the increase in slenderness ratios increases the lateral load capacity. When slenderness ratio changes from 25 to 38, the percentage increase in pile capacity is in the range of 7.57-7.90%, the percentage increase in pile capacity is quite low.

Figure 7, it is observed that when two pile group to nine pile group, the behavior of pile is almost like two pile group. It shows very clearly that if pile group decreased from nine piles to two piles, the effect of two pile group is almost small on the lateral load pile capacity. The experimental results are compared with those obtained from finite element analysis (FEA) SoilWorks 2D and found to be in good agreement.

Using the soil reaction-displacement (p - y) relationships the soil property values proposed in the standard: [14] p - y formulation, the predicted lateral response for the test pile is shown in comparison to the experimental response in Figures 7. For the bamboo pipe pile, the predicted and experimental lateral loads are in good agreement initially, but the numerical predicted lateral load is less than the experimental maximum lateral load.

5.2 Lateral Displacement, Bending Moment, Shear Force and Soil Reaction-Depth Behaviour of Piles

Figure 8(a) show that the deflected shapes of the pile are predicted reasonably well recommended: [14]. However, the predictions overestimate the lateral displacements for the experimental. This could be related to the characteristics of the soil-reaction (p-y) curves that may be on the “soft” side at low lateral load levels and “stiff” at high lateral load levels. Since there are several possible ways to connect these two states, it is conceivable that other shapes of transition zone will result in different prediction results. A transition zone shape that is stiffer at low lateral load level and softer at high lateral load levels may be better suited to capture the behaviour of the soils encountered at this work. However, the approach was to use default p-y model that would produce the best level of prediction possible.

From figure 8(b) it observed that, the maximum bending moment increases while decrease in the pile group. In addition, for the same length of pile (say 720mm), the bending moment is more in loose layer in-between dense layers for all pile group configurations. Here, because of the decrease in the resistance at the top portion of the soil mass as there is a reduction in the soil mass in the loose layer in-between dense layers. In addition that, the depth at which the maximum bending moment (MBM) occurs at depth of fixity, then decreases with increase in the flexural stiffness of the pile and the soil because of the increase in the embedded length of the pile. Since the bending moment profile, it observed that the depth of fixity occurs almost at a depth of 08.80d and 09.60d below the soil surface for slenderness ratios, 25 and 30 respectively. Here it observed that, there is little change in depth of fixity because of the changes in the soil layer.

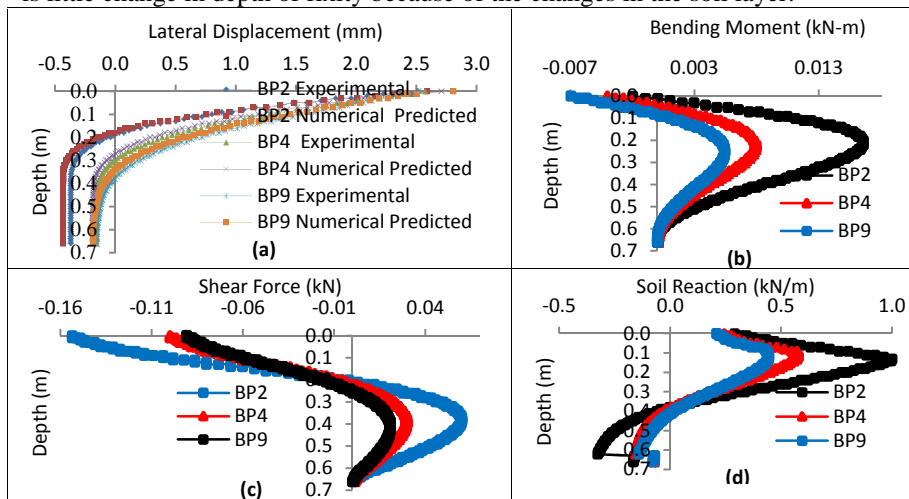


Fig.8. (a). Lateral displacement, (b). Bending moment, (c). Shear force and (d). Soil reaction v/s depth curves of bamboo pipe piles in loose layer in-between dense layers with slenderness ratio (SL) as 25.

Figure 8(c) show that, the depth at which the maximum shear force occurs at depth fixity, decreases with increase in the flexural rigidity (EI) of the pile and the soil because of the increase in the embedded length of the pile. Here this is also observed that, there is little change in depth of fixity because of the variation in soil layer. Note that, for each density, there is not a large difference between the maximum shear force for the different sizes (i.e., four pile to nine pile group).

Figure 8(d) present that, the pile in the uniform horizontal elastic medium, the soil reaction reduces along depth and has a maximum value at a depth 0.14m for slenderness ratio 25. Of course, this is in realistic. In a real situation, the soil below the ground surface will yield when its limiting shear strength is reached. Then the, soil reaction close to the depth of pile should be small.

6 CASE STUDY

The finite element modeling adopted in this study has been carried out on a selected laterally loaded bored R.C.C. pile at Falta, Sector-II, FEPZ, South 24 Parganas (S) at Kelambakkam site upto the explored dept of around 45.0m. below ground level. Cut-off length is considered as 2.0m. Data were taken during the lateral loading of three 450, 500 and 550mm diameter bored R.C.C.test piles installed at a site where the soils consisted of clean fine sand to silty fine sand with safe lateral load carrying capacity was 27.00kN to achieve target displacement of 5mm. The static loading was employed.

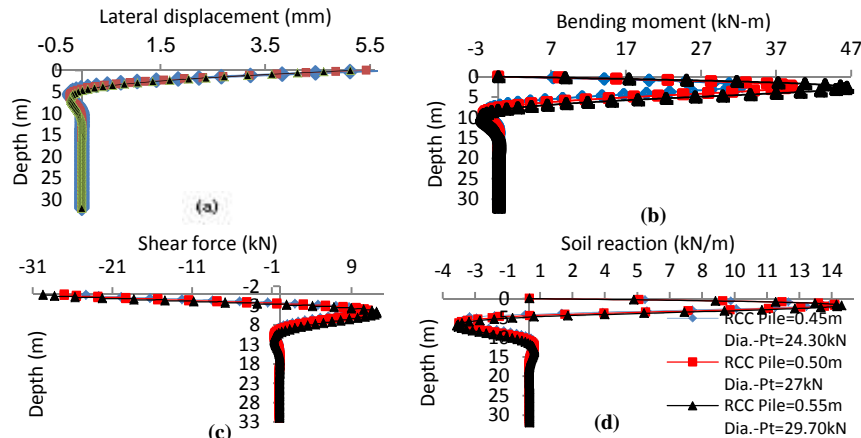


Fig.9. (a to d): Lateral displacement, Bending moment, Shear force and Soil reaction Vs depth with different diameters of 32m length RCC bored pile in in-situ layered soil.

From careful investigation of the illustrated results, it can be concluded that the different pile diameters has a notable effect on the pile response under lateral loading. Although the pile diameters slightly influence the predicted maximum pile bending moment, the initial modulus of subgrade reaction is considerably influenced by uncertainties in different pile diameters.

7 Conclusions

Following conclusions were drawn based on results obtained from p-y approach and numerical programme on laterally loaded piles in layered soil;

1. The proposed method for predicting p-y curves for laterally loaded piles in sand involves the use of the parameters which are believed to be important and employs available theories for predicting soil behaviour.
2. Predictions of the behaviour of the bamboo pipe piles, using p-y curves developed by the proposed method, agree very well with the experimental.
3. The effects of lateral load on the lateral resistance of pile groups depend on pile arrangement, flexural stiffness of pile material, and soil density.
4. It is observed that, lateral load displacement behaviour of the pile and pile group is non linear in most of the cases studied with different combination of number of piles, stiffness of the pile material and density of sand layers.
5. The increase in pile group configuration increases the lateral load capacity of the pile. For a pile group with slenderness (L/d) ratio 30 the percentage increase is 9.20% to 25.38%.

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