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Static Analysis of thin Rectangular Plate Resting on Pasternak foundation

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Abstract. Static analysis of four – nodded thin rectangular plate element based on Kirchhoff theory resting on Pasternak foundation. All the deformation stiffness matrix of plate and subsoil are evaluated using finite element method. A Matlab code is developed for present formulation then convergence study is carried out, then validation done and then the static analysis of thin plates resting on Pasternak type foundations. The results, thus obtained, are compared, with the available results obtained by other researchers. Parametric study done and the maximum deflection, bending moment are presented in tabular and graphical forms. It is concluded that the effect of the soil coefficient on the static analysis of the plates on elastic foundation is generally larger than that of the aspect ratio. It behaves extremely well for thin plate, results are very close to exact solution and convergence rate is high.

Key Words: Pasternak foundation, finite element, Shear parameter and sub-grade reaction**.**

1 Introduction

Analyses of plates on elastic foundations have wider applications in aerospace, civil and mechanical engineering. Developing more realistic foundation models and simplified methods to solve this complex soil-structure interaction problem are very important for safe and economical design. Majority of the problems cannot be solved by theoretical approach, led use the numerical techniques like sophisticated finite element method.

In static analysis of plates resting on the elastic foundation using the Winkler model, a single parameter model neglects the shear deformations between closely spaced elastic springs. Winkler model main discrepancies are the discontinuity in the soil displacement between the soil under the structure and that outside the structure. To overcome the discrepancies of Winkler model two-parameter foundation models developed by Hetenyi [1], Filonenko Borodich [2] and Pasternak [3] provide for the displacement continuity of the soil medium by adding of a second spring which interacts with the first spring of the Winkler model. The mechanical modelling of plate-subsoil interaction problem is mathematically quite complex phenomenon and the response of sub-grade is depending upon by many factors. The analysis of plates resting on elastic foundations is the great interest of researcher and vast area of various research studies. Mishra and Chakrabarti [4] investigated shear and contact effects on the behaviour of rectangular plates resting on tensionless elastic foundation using finite element method. They used a nine-nodded Mindlin plate element to account for transverse shear effects. Buczkowski and Torbacki [5], developed an 18-node isoparametric interface element of zero-thickness that account for shear deformation of the plate, details analyzed thick plates resting on two-parameter elastic foundation. Celik and Saygun [6] developed a finite element formulation for plates on elastic foundation incorporating the shear deformations in the behaviour of the plate, and the effect of subsoil is considered as a combination of elastic bending and shear deformation of the soil. Daloglu and Ozgan [7] developed an iterative method to determine the subsoil depth affected from the load on the plate resting on elastic foundation using stress distribution within the subsoil depending on the loading and dimension of the plate. Ozgan and Daloglu [8] investigate in details the effect of transverse shear strains on thin and the thick four-nodded and eight-nodded Mindlin plate resting on elastic foundation using modified Vlasov model. Turhan [9] studied in details thin plate resting on elastic foundation using modified Vlasov model using FEM. W.T. Straughan [10] studied in details thin plate resting on elastic foundation using modified Vlasov model using FDM. Ozgan and Daloglu [11] investigate in details the effect of shear strain on thick plate using a four-nodded and an eight-nodded plate bending element based on Mindlin plate theory have been adopted for modeling the thick plates on elastic foundations using Winkler model incorporate transverse shear deformations. Buczkowski and Torbacki [12] developed the finite element technique that account for the material properties of soil and incorporate the surrounding effect outside the plate.

In the present paper, a four-nodded thin rectangular plate resting on Pasternak foundations for static analysis using finite element method. For numerical integration Gauss-Legendre-type quadrature rule is used. This allows computation to be more accurate than other quadrature rule. Convergence rate, accuracy and applicability of the present formulation for static analysis of thin plate on Pasternak foundation are demonstrated through number of numerical examples.

2 Methodology

2.1 Pasternak model

In this model, existence of shear interaction among the spring elements is assumed which is accomplished by connecting the ends of the springs to a plate that only undergoes transverse shear deformation. The load–deflection relationship is obtained by considering the vertical equilibrium of a shear layer. In this two-parameter model, the reaction of the foundation is determined by a vertical spring constant, k in combination with a shear parameter Gb, which interacts with the vertical springs. The values of k and Gb parameter can be determined by using a methodology given by Selvadurai (1979) [13]

$$k = \frac{E_s(1 - v_s)}{H(1 + v_s)(1 - 2v_s)}$$

$$\text{and } Gb = \frac{E_s H}{6(1+v_s)}$$

E_s, ν_s = Young's modulus of elasticity and Poisson's ratio of soil.
H is the thickness of soil which corresponds to the end of the influence zone for the foundation based on Boussinesq's method, H = 2B where B is the width of plate.
Therefore the field equation in the domain, Ω, can be written as

$$D\nabla^4 w + kw - Gb\nabla^2 w = q \quad (1)$$

$$\nabla^2 = \text{Laplacian operator} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\,\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$$

2.2 Finite element formulation

It has four corner nodes and each node is associated with three degrees of freedom.

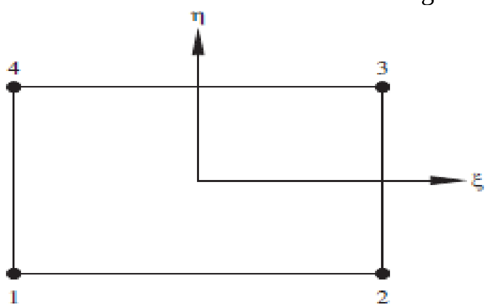

Fig. 1. PBR4 plate elements

Length and width of the plate are 2a and 2b respectively.

$$u(x,y,x) = -z\frac{\partial w_0}{\partial x}; v(x,y,z) = -z\frac{\partial w_0}{\partial y}; \text{and } w(x,y,z) = w_0(x,y).$$

It is further assumed that as, w(x, y, 0) = w(x,y).
The displacements of the surface of the soil are equal to the displacements of the middle surface of the plate.

$$\text{The nodal displacement at i}^{\text{th}} \text{ node } \{\delta_i\} = \left\{ w_i \quad \left(\frac{\partial w}{\partial x}\right)_i \quad \left(\frac{\partial w}{\partial y}\right)_i \right\}^T$$

The element displacement vector is defined as $\{d_e\} = \{d_1\ d_2\ d_3\ d_4\}^T$ for four nodded elements.
The element is based on thin plate theory. Hence, it is sufficient to prescribe variation of transverse displacement w on element region.
The strain displacement relation for plane stress condition is given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w_0}{\partial x^2} = -z\chi_x;\ \varepsilon_{yy} = \frac{\partial v}{\partial y}; \varepsilon_{yy} = -z\frac{\partial^2 w_0}{\partial y^2} = -z\chi_y; \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{xy} = -z\left(\frac{\partial^2 w_0}{\partial xdy} + \frac{\partial^2 w_0}{\partial xdy}\right) = -2z\frac{\partial^2 w_0}{\partial xdy}; \gamma_{xy} = -z\chi_{xy} \text{ and } \varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$$

Hence, generalized curvatures are written as

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 \mathrm{w}_0}{\partial \mathrm{xdy}} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 N}{\partial x^2} \\ \frac{\partial^2 N}{\partial y^2} \\ 2\frac{\partial^2 \mathrm{N}}{\partial \mathrm{xdy}} \end{bmatrix} \{\mathrm{w}_0\};\ \therefore\ [B_{bi}] = \begin{bmatrix} \frac{\partial^2 N}{\partial x^2} \\ \frac{\partial^2 N}{\partial y^2} \\ 2\frac{\partial^2 \mathrm{N}}{\partial \mathrm{xdy}} \end{bmatrix};$$

$[B_b] = [B_{b1}\ B_{b2} \ldots\ldots\ldots B_{b12}]$ for four nodded elements.

Stresses are related to strain in terms of elasticity matrix for plane stress case is given by

$$C_{11} = \frac{E}{(1-\nu^2)}\ and\ G = \frac{E}{2(1+\nu)}; C_{22} = C_{11}; C_{33} = C_{11}; C_{12} = \nu C_{11}; C_{13} = C_{12};$$

$$C_{21} = C_{12}; C_{23} = C_{12}; C_{31} = C_{12}; C_{32} = C_{12}; C_{44} = G; \therefore\ [C_b] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{44} \end{bmatrix};$$

$$[D_\mathrm{b}] = \int_{-\frac{\mathrm{h}}{2}}^{\frac{\mathrm{h}}{2}} \mathrm{z}[\mathrm{C_b}]\, dz\ ; [\mathrm{D_b}] = \frac{\mathrm{Eh}^3}{12(1-\mathrm{v}^2)} \begin{bmatrix} 1 & \mathrm{v} & 0 \\ \mathrm{v} & 1 & 0 \\ 0 & 0 & \frac{1-\mathrm{v}}{2} \end{bmatrix}$$

$[D_b]$ is the Plate rigidity matrix.
E = the modulus of elasticity of the plate, h = thickness of the plate, and
ν = Poisson's ratio of the plate.,
Hence bending stiffness matrix

$$[K_b] = \int_{-1}^{1}\int_{-1}^{1} [B_b]^T [D_b][B_b]|J| ds dt \qquad (2)$$

Following usual steps, the bending is expressed as

$$[K_b] = \sum_{\mathrm{j}=1}^{2}\sum_{i=1}^{2} W_i W_\mathrm{j} |J| [B_b]^T [D_b][B_b]$$

From these equations, it is observed that full 2 × 2 Point Gauss-Legendre type quardature is adopted for bending stiffness.
Considering a structural element which has a differential area 'dA' in contact with the foundation the lateral deflection of area 'dA' normal to the foundation is, w= $[N_f]\{d\}$
The strain energy U_r in a linear spring is given by eq. $= \frac{1}{2} kw^2$

$U_r = \frac{1}{2}\int kw^2\, dA;$ k is the soil parameter known as modulus of sub-grade reaction.

$As\ `w'\ is\ a\ scalar, so\ w^2 = w^T w; w^2 = \{d\}^T [N_f]^T [N_f]\{d\}$

$$Strain\ energy\ U_r = \frac{1}{2}\int k\{d\}^T [N_f]^T [N_f]\{d\}\, dA = \frac{1}{2}\{d\}^T [K_f]\{d\}$$

In which the foundation stiffness matrix for the element is, $[K_f] = \int k[N_f]^T [N_f]\, dA$

$$[K_f] = kab\int_{-1}^{1}\int_{-1}^{1} [N_f]^T [N_f]\, ds dt k \int_{-1}^{1}\int_{-1}^{1} [N_f]^T [N_f]\, |J| ds dt \qquad (3)$$

If the problem deals with a plate on elastic foundation, $[N_f]$ is identical to the shape function matrix [N] of the plate.

A typical sub-matrix for foundation parameter corresponding to i - th node is

$$[K_{fi}] = k\sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j |J| N_i^T N_j \,; \therefore [K_f] = \{K_{f1}\, K_{f2} \dots\dots\dots\dots K_{f12}\}^T \tag{4}$$

Strain energy stored by foundation parameter 'Gb' is given by

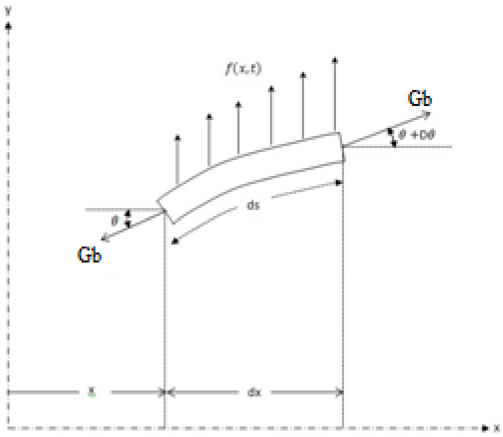

Fig.2. Equilibrium condition of differential beam element under foundation second parameter

The length of a differential element 'dx'in the deformed position, 'ds', can be expressed as

$$ds = \sqrt{(dx)^2 + (dw)^2} = dx\sqrt{1 + \left(\frac{dw}{dx}\right)^2}$$

$$ds = dx\left[1 + \left(\frac{dw}{dx}\right)^2\right]^{\frac{1}{2}} = dx\left[1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$\therefore ds - dx = \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

$$U = \frac{1}{2}\int_{-a}^{a} \mathrm{Gb}\left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{1}{2}\int_{-a}^{a} \mathrm{Gb}\{d\}^T [N_f']^T [N_f']\{d\}\, dx = \frac{1}{2}\{d\}^T [K_{ex}]\{d\}$$

$$where\ [K_{ex}] = \int_{-a}^{a} \mathrm{Gb}[N_f']^T [N_f'] dx;\ S = [N_f'];$$

$$\text{Similarly for Y} - \text{direction where } [K_{ey}] = \int_{-b}^{b} \mathrm{Gb}[N_f']^T [N_f'] dy\,; R = [N_f']$$

$$\therefore Stiffness\ for\ shear\ parameter, \mathrm{Gb}\ is\ [K_e] = \mathrm{Gb}\iint [S^T S + R^T R] dA$$

$$[K_e] = \mathrm{Gb}ab\int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{a^2}\left[\frac{\partial N}{\partial s}\right]^T \left[\frac{\partial N}{\partial s}\right] + \frac{1}{b^2}\left[\frac{\partial N}{\partial t}\right]^T \left[\frac{\partial N}{\partial t}\right]\right) dsdt$$

A typical sub-matrix for second foundation parameter corresponding to i-th node is

$$[K_{e_i}] = Gb\left(\sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j |J| \frac{dN}{dx}_i + \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j |J| \frac{dN}{dy}_i\right)$$

For four nodded elements, $[K_e] = [K_{e1}\ K_{e2} \dots\dots\dots . . K_{e12}\ ]$ (5)
Stiffness matrix of the elements $[K] = [K_b] + [K_f] + [K_e]$ (6)
The element load vector for a plate due to transverse distributed load of q per unit area acting top of the plate

$$\{f\} = q \int_{-1}^{1} \int_{-1}^{1} [N]^T \, |J| ds dt$$

$. Force\ vector\ for\ a\ point\ load\ \{f\} = [N_d]^T [F]$

The element load matrix for n nodded plate due to transverse distributed load of q per

$$unit\ area\ acting\ top\ of\ the\ plate \{q_i\} = q \sum_{i=1}^{2} \sum_{j=1}^{2} W_i W_j |J| N_i$$

Integration is carried out using 2 × 2 Gauss-Legendre integration.
For four nodded elements, $\{f\} = \{q_1\ \ q_2 \dots\dots\dots\dots . q_{12}\}^T$ (7)
Similarly a typical sub-matrix for foundation parameter corresponding to i - th node is
$Hence\ obtain\ \ [x] = [K]^{-1}\{f\}$ (8)

3 Result and discussion

3.1 Convergence study and test the formulation

To ease comparison of results, the following dimensionless parameters are defined:
1 Non dimensional maximum deflection $\bar{w}$ =1000Dw/$q_o a^4$; Maximum moment M=100M/$q_o a^2$ for uniformly distributed load, q_o and deflection $\bar{w}$ =1000Dw/Pa^2; Maximum moment M=100 M/P for point load, P;
2. Foundation parameter, Kw = $(Ka^4/D)^{1/4}$;Ks = $(Gba^2/D)^{1/2}$.
A square plate is considered first to test the present formulation and simultaneously a convergence study is performed taking Kw and Ks both 5. The mesh size of 14 × 14 is decided for a reasonable result.

3.2 Validation work

This chapter starts with some comparisons with similar studies done by other researchers are made. The present method can be applied to static analysis of plate with various boundary conditions, including free-free.
One example has been chosen from the study done by Ömer CİVALEK [14] for validation of the present formulation. A square plate for verification were taken h/a = 0.005 and non-dimensional value of Kw^4 = 200 and Ks^2 = 5, 10, 20. The non-dimensional central deflections and bending moment for SSSS plate with uniform loading listed in table 1 and have been compared with Ömer CİVALEK [14] show excellent agreement.

Table 1. Comparison of non-dimensional central deflections and bending moment in isotropic square plate subjected to uniform loading and resting on Pasternak foundation.

a/b = 1 , a/h = 200 FULL UDL B.C. - SSSS, ν = 0.25				
THEORY	Kw^4	Ks^2	w (a/2,b/2)	M (a/2,b/2)
PS	200	5	2.2702	2.3789
Ömer CİVALEK [19]			2.2640	2.4208
PS		10	1.9777	2.0474
Ömer CİVALEK [19]			1.8860	1.9876
PS		20	1.5705	1.5918
Ömer CİVALEK [19]			1.5700	1.6133

3.3 Parametric study

For parametric study the data has been taken as a/b = 1, 1.5, 2 and a/h =100 and Kw = 3, 5, 7 Ks = 3, 4, 5. For limited space here present some of them.
Figure 3, 4 shows the variation of maximum non-dimensional displacement parameter and maximum non-dimensional bending moment parameter with aspect ratio. It is observed from figure that the maximum displacement parameter and maximum bending moment parameter both decreases with increase of aspect ratio but it shows that aspect ratio greater than 1.5 maximum displacement parameter and maximum bending moment parameter both tends to stable with increase of aspect ratio.

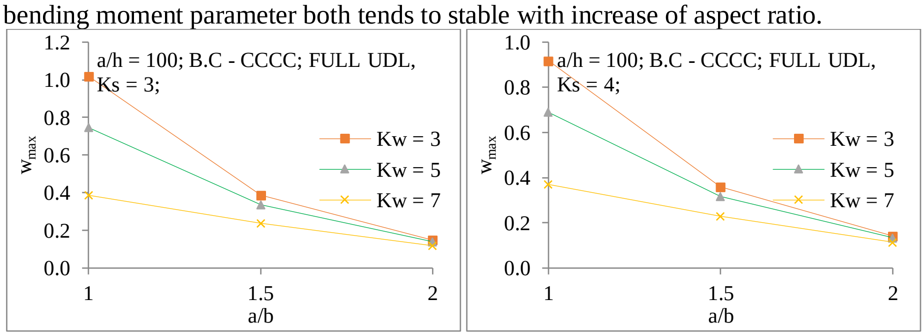

Fig. 3. The effect of aspect ratio on non-dimensional deflection of clamped plate subjected to UDL.

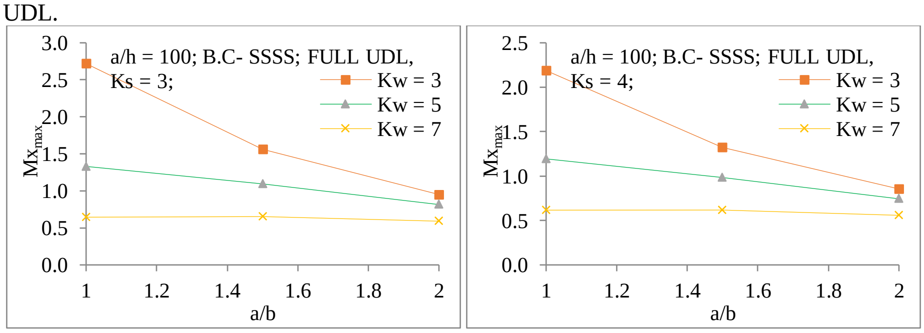

Fig. 4. The effect of aspect ratio on non-dimensional moment of simply supported plate subjected to UDL.

Figure 5, 6 shows the variation of maximum displacement parameter and maximum bending moment parameter with foundation parameter in case of both uniform loading and point load. It is observed from figure that the maximum displacement parameter and bending moment parameter both Mx and My decreases with increases of foundation parameter but in case of free plate with point load is concave and other cases it is more or less convex.

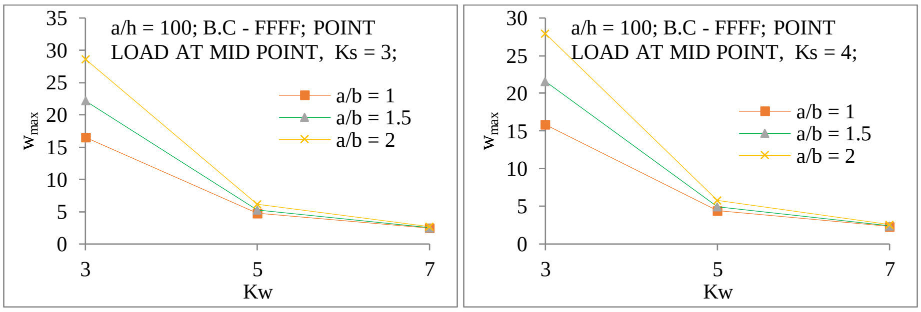

Fig. 5. The effect of foundation modulus on non-dimensional deflection of free plate subjected to point load at mid point.

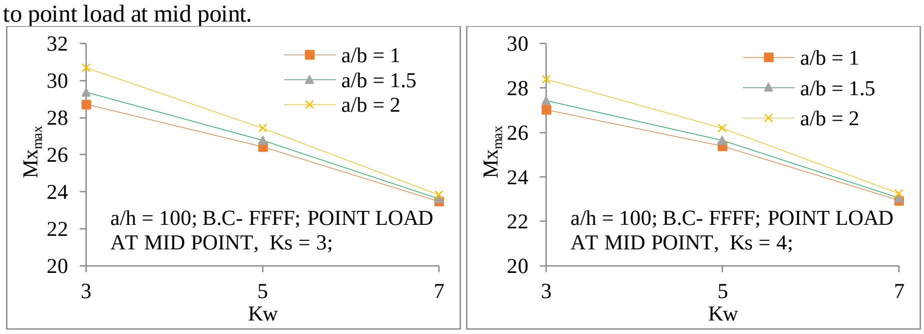

Fig. 6. The effect of foundation modulus on non-dimensional moment of free plate subjected to point load at mid point.

From Figure 2-5 it is observed that the maximum displacement parameter and maximum bending moment parameter decreases with increase of constrains on the edges this shows that higher constraints on the edges increase the flexural rigidity of the plate and hence lower displacement and bending moment.

4 Conclusions

The accuracy and the efficiency of the element for different foundation parameter and different load cases and then a parametric study are performed. It is seen that this rectangular plate bending elements can be used effectively and efficiently for analyzing thin plates on elastic foundations under any type of load cases and boundary condition. The effect of foundation parameter on the displacement is larger for concentrated load case than for distributed load case, and this effect increases as h/a ratio increases for any foundation parameter. The observations indicate that the effect of aspect ratio of plate on the behaviour of the plate bending is always larger for free plates.

The presented examples show some of the advantages of the suggested approach for numerical solution of a plate on an elastic foundation. It gives opportunities for:

1. Application of various loads at an arbitrary Point or a region on the plate;
2. The approach can be performed on a thin plate effectively and efficiently;
3. The plate and the soil medium stiffness can vary smoothly along the plate's length;

References

1. Hetenyi M. A general solution for the bending of beams on an elastic foundation of arbitrary continuity. J Appl Phys 1950; 21:55–8.
2. Filonenko-Borodich, M.M. (1940) some approximate theories of elastic foundation, Uchenyie Zapiski Moskovskogo Gosudarstuennogo Universiteta Mekhanika, Moscow, 46, pp 3-18 (in Russian).
3. Pasternak, P.L. (1954) on a new method of analysis of an elastic foundation by means of two foundation constants, Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstuve i Arkhitekture, Moscow (in Russian).
4. Mishra RC, Chakrabarti SK. Shear and attachment effects on the behaviour of rectangular plates resting on tensionless elastic foundation. Eng Struct 1997;19(7):551–67.
5. Buczkowski R, Torbacki W. Finite element modeling of thick plates on two parameter elastic foundation. Int J Numer Anal Meth Geomech 2001;25: 1409–27.
6. Celik M, Saygun AA. Method for the analysis of plates on a two-parameter foundation. Int J Solids Struct 1999; 36:2891–915.
7. Daloglu AT, Ozgan K. The effective depth of soil stratum for plates resting on elastic foundation. Struct Eng Mech 2004; 18(2):263–76.
8. K. Ozgan, A.T. Daloglu. Effect of transverse shear strains on plates resting on elastic foundation using modified Vlasov model. Thin-Walled Structures, 46: 1236–1250, 2008.
9. Turhan A. A consistent Vlasov model for analysis of plates on elastic foundations using the finite element method. PhD. dissertation, Graduate School of Texas Tech. University, Lubbock, TX, 1992.
10. Straughan WT Analysis of plates on elastic foundation. PhD. dissertation, Graduate School of Texas Tech. University, 1990.
11. K. Ozgan, A.T. Daloglu. Alternative plate elements for the analysis of thick plates on elastic foundation. Structural Engineering and Mechanics, 26(1): 69–86, 2007.
12. R. Buczkowski, W. Torbacki. Finite element analysis of plates on layered tensionless foundation. Archives of Civil Engineering, 56(3): 255–274, 2010.
13. Selvaduari APS. Elastic analysis of soil-foundation interaction. Amsterdam: Elsevier; 1979.
14. Ömer CİVALEK, static analysis of shear deformable rectangular plates on winkler-pasternak foundation, Journal of Engineering and Natural Sciences, Sigma Vol. 25 Issue 4.